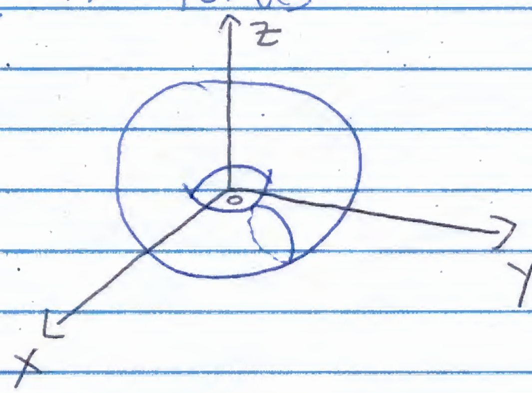
Topic: ∘ Surface Area

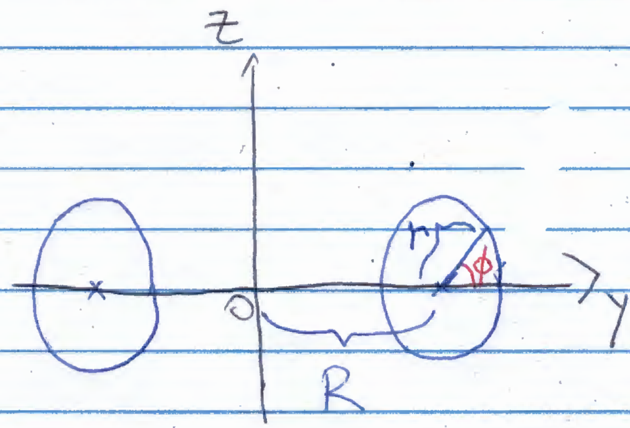
Surface Area:

- Let S be a surface in $\mathbb{R}^3$ parametrized by $\Sigma : D \to \mathbb{R}^3$, where Σ is regular.
Then the area of S is given by the formula

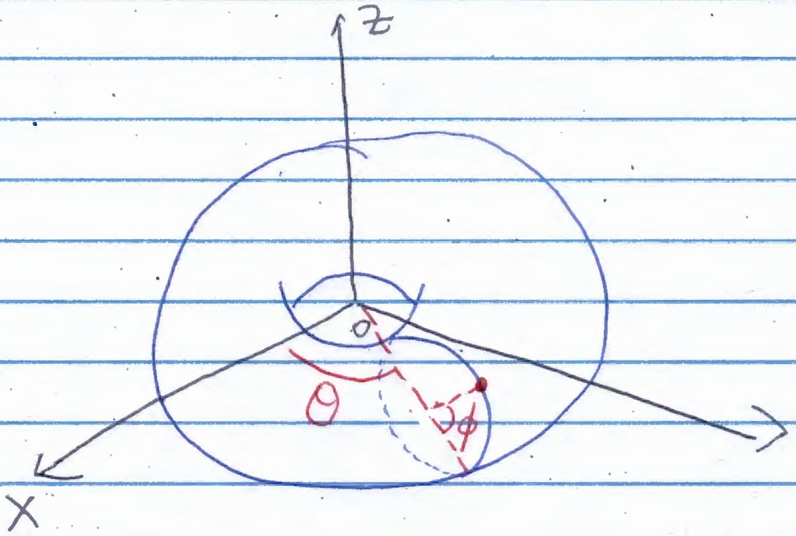
$$\text{Area}(S) = \iint_D \|\Sigma_u \times \Sigma_v\| \, dA.$$

Example: 1) Torus.





(I) Parametrization of torus.



$$\Sigma(\theta, \phi) = \left((R + r\cos\phi)\cos\theta, (R + r\cos\phi)\sin\theta, r\sin\phi \right),$$

where $\theta, \phi \in [0, 2\pi]$.

(II) Find $\|\mathbf{X}_\theta \times \mathbf{X}_\phi\|$.

$$\mathbf{X}_\theta \times \mathbf{X}_\phi = \begin{vmatrix} i & j & k \\ -(R+r\cos\phi)\sin\theta & (R+r\cos\theta)\cos\theta & 0 \\ -r\sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi \end{vmatrix}$$

$$= \big(r(R+r\cos\phi)\cos\theta\cos\phi,\ r(R+r\cos\phi)\sin\theta\cos\phi,\ r(R+r\cos\phi)\sin\phi \big)$$

$$\therefore \ \|\mathbf{X}_\theta \times \mathbf{X}_\phi\| = r(R+r\cos\phi) \neq 0.$$

(III) Area of torus.

$$\text{Area of torus} = \iint_D \|\mathbf{X}_\theta \times \mathbf{X}_\phi\|\, dA$$

$$= \int_0^{2\pi}\int_0^{2\pi} r(R+r\cos\phi)\, d\phi\, d\theta$$

$\left(\int_0^{2\pi} \cos\phi\, d\phi = 0\right) \rightarrow$

$$= 2\pi \int_0^{2\pi} rR\, d\phi$$

$$= 4\pi^2 r^2 R \,_{''}$$

2) Graph of a differentiable function.

- Let S be the graph of the differentiable function $f: D \subset \mathbb{R}^2 \to \mathbb{R}$, where D is the domain of f.

<u>Goal</u>: Find Area(S).

Note that S is parametrized by

$$\Phi(x,y) = (x, y, f(x,y))$$

$$\Phi_x = (1, 0, f_x), \quad \Phi_y = (0, 1, f_y)$$

$$\Phi_x \times \Phi_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$= (-f_x, -f_y, 1) \neq 0.$$

$$\Rightarrow \|\Phi_x \times \Phi_y\| = \sqrt{1+f_x^2+f_y^2}$$

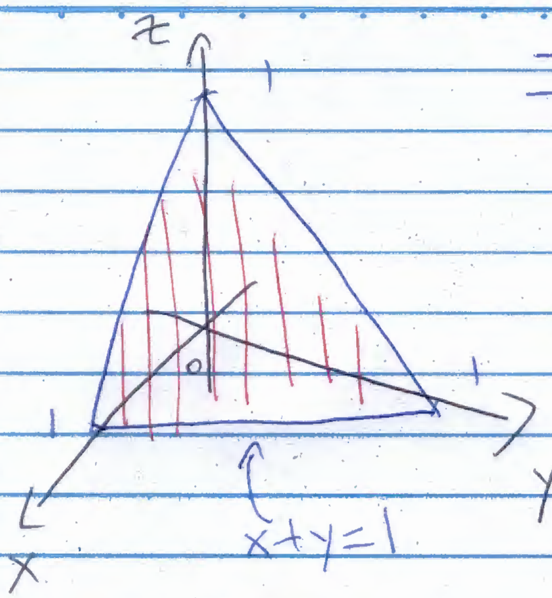
$$\therefore \text{Area}(S) = \iint_D \sqrt{1+f_x^2+f_y^2}\, dA.$$

<u>Examples</u>: i) Find the surface area of the part $x+y+z=1$ that lies in the first octant.

<u>Ans</u>: Parametrize S by

$$\Phi(x,y) = (x, y, 1-x-y)$$

$$\text{Area}(S) = \iint_D \sqrt{1+f_x^2+f_y^2}\, dA$$



$$= \int_0^1 \int_0^{1-x} \sqrt{1+1+1}\, dy\, dx$$

$$= \frac{\sqrt{3}}{2} \quad //$$

ii) Find the surface area of the part $z = xy$ that lies in the cylinder given by $x^2 + y^2 = 1$.

<u>Ans:</u> Parametrize S by

$$\Phi(x,y) = (x, y, xy)$$

$$\text{Area}(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

(Polar coordinate)

$$= \iint_{\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}} \sqrt{1 + x^2 + y^2}\, dA$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1+r^2}\, dr\, dr\, d\theta$$

$$= 2\pi \left(\frac{1}{2}\right) \int_0^1 \sqrt{1+r^2}\, d(1+r^2)$$

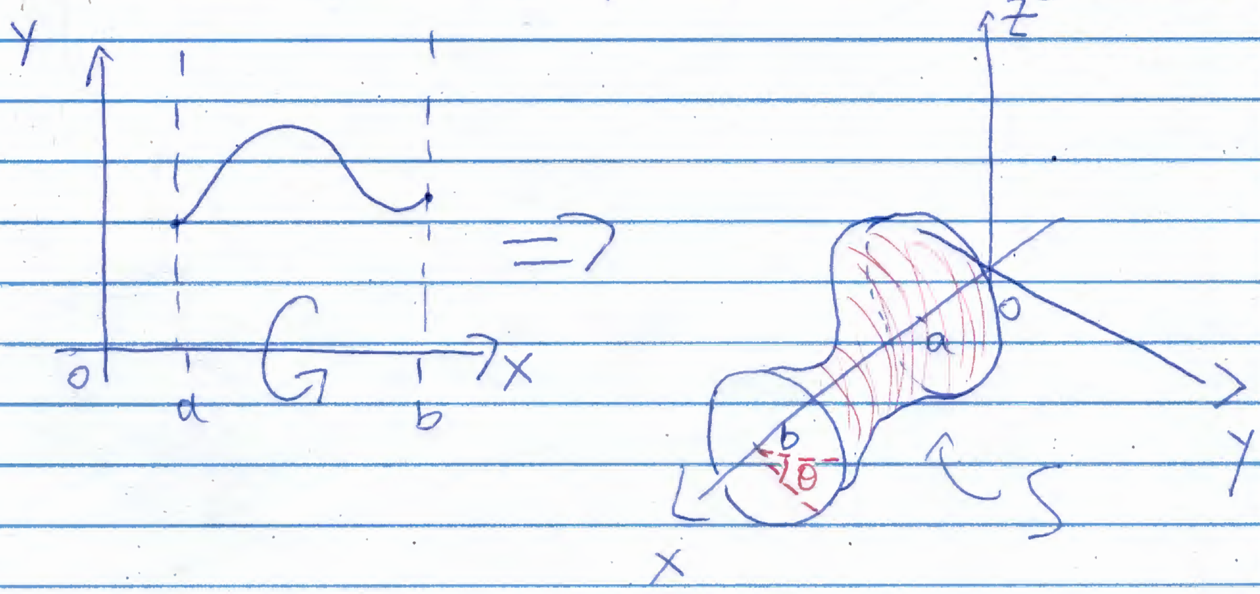
$$= \pi \left[\frac{2}{3} (1+r^2)^{\frac{3}{2}} \right]_0^1$$

$$= \frac{2}{3}\pi (2\sqrt{2}-1)$$ //

3) Surface of revolution:

- Let $f: [a,b] \Rightarrow \mathbb{R}^+$ be a differentiable function.

  Let S be the surface obtained by revolving the graph $y=f(x)$ along x-axis.



Parametrize S by

$$\Phi(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$$

$$\Phi_x \times \Phi_\theta = \begin{vmatrix} i & j & k \\ 1 & f'\cos\theta & f'\sin\theta \\ 0 & -f\sin\theta & f\cos\theta \end{vmatrix}$$

$$= (ff', -f\cos\theta, -f\sin\theta)$$

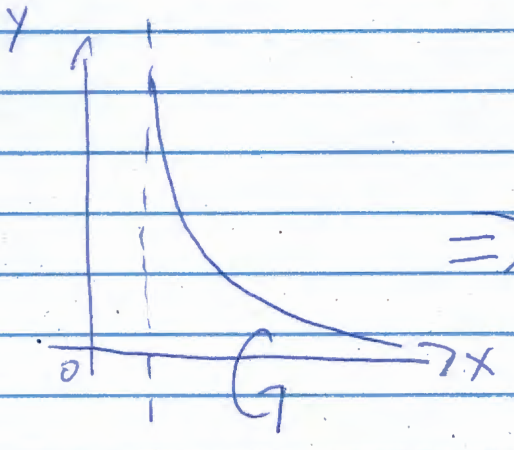
$$\therefore \|\mathbb{X}_x \times \mathbb{X}_\theta\| = \sqrt{(ff')^2 + f^2} = f\sqrt{1+(f')^2}$$

$$\therefore \text{Area}(S) = \iint_D f\sqrt{1+(f')^2}\, dA$$

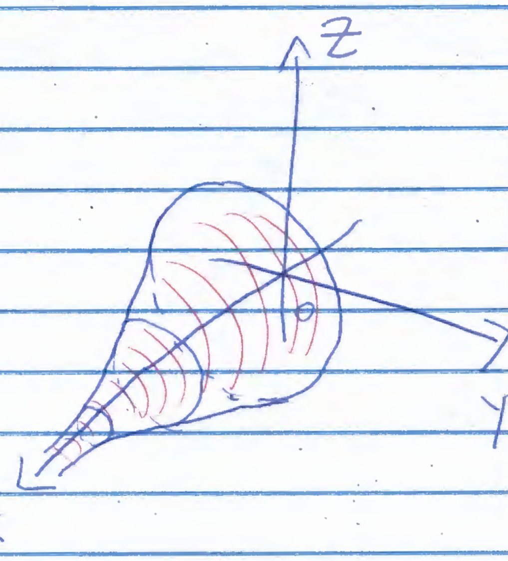
$$= \int_0^{2\pi}\int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx\, d\theta$$

$$= 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx$$

Example: i) Find the area and the underlying volume of the surface obtained by revolving the graph $y = \frac{1}{x}$ for $x \geq 1$



$\Rightarrow$



Ans: $\text{Area}(S) = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\, dx$

$$\geq 2\pi \int_1^\infty \frac{1}{x}\, dx$$

$$= 2\pi \left[\ln x\right]_1^{\infty}$$

$$= \lim_{x \to \infty} (2\pi \ln x)$$

$$= +\infty$$

$$\text{Volume}(S) = \iiint_R dx\,dy\,dz$$

$$\overset{\text{Fubini's}}{\underset{\text{thm}}{=}} \int_1^{\infty} \left(\iint_{\{(y,z) \mid y^2+z^2 \le \frac{1}{x^2}\}} dA \right) dx$$

(Area of disk with radius $\frac{1}{x}$)

$$= \int_1^{\infty} \pi \left(\frac{1}{x^2}\right) dx$$

$$= \pi \left[-\frac{1}{x}\right]_1^{\infty}$$

$$= \pi \quad //$$

Remark: This gives a surface with infinite surface area but finite underlying volume.

4) Let S be a surface parametrized by $\mathbf{X}(u,v)$ on a domain $D \subseteq \mathbb{R}^2$. At each point $(u_0, v_0) \in D$, denote the angle between $\mathbf{X}_u$ & $\mathbf{X}_v$ by θ.

$$\Rightarrow \|\mathbf{X}_u \times \mathbf{X}_v\|^2 = \|\mathbf{X}_u\|^2 \|\mathbf{X}_v\|^2 \sin^2\theta$$

$$= \|\mathbf{X}_u\|^2 \|\mathbf{X}_v\|^2 - \|\mathbf{X}_u\|^2 \|\mathbf{X}_v\|^2 \cos^2\theta$$

$$\Rightarrow \|\Phi_u \times \Phi_v\|^2 = \|\Phi_u\|^2\|\Phi_v\|^2 - \langle \Phi_u, \Phi_v\rangle^2.$$

$$\therefore \text{Area}(S) = \iint_D \|\Phi_u \times \Phi_v\|\, dA$$

$$= \iint_D \left(\|\Phi_u\|^2\|\Phi_v\|^2 - \langle \Phi_u, \Phi_v\rangle^2 \right)^{\frac{1}{2}} dA$$

$$= \iint_D \sqrt{EG - F^2}\, dA,$$

where $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \Phi_u, \Phi_u\rangle & \langle \Phi_u, \Phi_v\rangle \\ \langle \Phi_v, \Phi_u\rangle & \langle \Phi_v, \Phi_v\rangle \end{pmatrix}$

is called the first fundamental form,

with $\det(\text{1st fundamental form}) = EG - F^2$.